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# SMALL-PARAMETER METHOD FOR CONSTRUCTING APPROXIMATE STRATEGIES IN A CLASS OF DIFFERENTIAL GAMES

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We examine a class of problems in which the pay-off is some function of the terminal state of a conflict-controlled system. When the opportunities of one of the players are small in relation with the opportunities of the other, we propose methods for constructing approximate optimal strategies of the players, based on solving the Bellman equation containing a small parameter. We have shown that the players' approximate optimal strategies can be constructed if the solutions of the corresponding optimal control problems are known. The error bounds for the methods are proved and examples are considered. The arguments used rely on the results in [1—6] on the theory of differential games and on [7—11] devoted to optimal control synthesis methods for systems subject to random perturbations of small intensity.

**1. Statement of the problem.** Let the motion of a conflict-controlled system be described by the nonlinear equation

$$\frac{dx}{dt} = F(x, t, u, v), \quad u \in P, \quad v \in Q_\varepsilon, \quad x[t_0] = x_0, \quad t \in [t_0, T] \qquad (1.1)$$

Here $x$ is an $n$-dimensional vector, $u$ and $v$ are $r$-dimensional control vectors of the first and second players, respectively, $P$ and $Q_\varepsilon$ are closed bounded sets, $F$ is a continuous function satisfying a Lipschitz condition in $x$ and $v$. The pay-off is the quantity $f[x(T)]$ determined at the terminal instant $t = T$ in the position $x(T)$ realized. The first player tries to minimize $f[x(T)]$ under the most unfavorable behavior of the second player. The second player's task is to guarantee the game's completion with the largest possible value of the pay-off. We assume that the opportunities of one of the players are small in comparison with the opportunities of the other. Namely, we assume that the set $Q_\varepsilon$ can be contained within an $r$-dimensional sphere of radius $\varepsilon$ small in relation to the minimal radius of the sphere which can contain set $P$. We assume, further, that the right-hand side of system (1.1) satisfies the conditions from [1]

$$|x'F(x, t, u, v)| \leqslant \lambda(1 + \|x\|^2), \ u \in P, \ v \in Q_\varepsilon, \ \lambda = \text{const} \quad (1.2)$$

Function $u(x, t)$ which for any possible position $(x, t)$ sets in correspondence a closed set $u(x, t) \subset P$ is called the strategy of the first player [2, 3]. Set $u(x, t)$ is assumed to be upper semicontinuous with respect to the inclusion at $(x, t)$. The class of functions $v(x, t)$ which specify admissible strategies of the second player is similarly defined.

**2. Iteration method.** We consider the class of differential games described in Sect. 1 and having a saddle point. The following result is valid ([4], Theorem 3).

Lemma. Let there exist a continuously differentiable function $S(x, t)$ which for all $x$ and $t$ satisfies the boundary-value problem

$$\begin{gathered} S_t + \min_u \max_v \{(F(x, t, u, v), \ S_x)\} = S_t + \\ \max_v \min_u \{(F(x, t, u, v), \ S_x)\} = 0 \\ S(x, T) = f(x), \quad u \in P, \quad v \in Q_\varepsilon \end{gathered} \quad (2.1)$$

Here $S_x = (S_{x_1}, \ldots, S_{x_n})$ is the vector of first partial derivatives with respect to $x$, $S_t$ is the partial time derivative. Then, the set $u^*(x, t)$ of vectors $u^*$ and the set $v^*(x, t)$ of vectors $v^*$, providing the minimax and the maximin in (2.1) are such that the strategies $u^*(x, t)$ and $v^*(x, t)$ are the minimax and the maximin strategies of the first and second players, respectively; the strategy pair $(u^*(x, t), v^*(x, t))$ provides the saddle point of the game being examined.

Introducing the notation

$$\begin{gathered} \min_u \max_v \{(F(x, t, u, v), S_x)\} = \max_v \min_u \{(F(x, t, u, \\ v), S_x)\} = H(x, t, u^*, v^*, S_x) \\ u \in P, \ v \in Q_\varepsilon \end{gathered} \quad (2.2)$$

we construct the following iteration process.

Selecting a strategy $v^\circ(x, t) \equiv 0$ we consider the boundary-value problem

$$S_t^\circ + \min_u \{(F(x, t, u, v^\circ), \ S_x^\circ)\} = 0, \ S^\circ(x, T) = f(x), \ u \in P \quad (2.3)$$

Assuming that the solution of this problem exists and is continuously differentiable we find the strategy $u^\circ(x, t)$ from the condition

$$\min_u \{(F(x, t, u, v^\circ), \ S_x^\circ)\} = H(x, t, u^\circ, v^\circ, S_x^\circ), \ u \in P \quad (2.4)$$

A new strategy $v^1(x, t)$ for the second player is found from the condition

$$\max_v \{(F(x, t, u^\circ, v), \ S_x^\circ)\} = H(x, t, u^\circ, v^1, \ S_x^\circ), \quad v \in Q_\varepsilon \quad (2.5)$$

The strategy $u^\circ(x, t)$ defined in (2.3) and (2.4) is called the first-player's approximate strategy in the zero approximation. The strategy $v^1(x, t)$ is called the second player's approximate strategy in the first approximation. The next step of the iteration process is to solve the boundary-value problem

$$S_t^1 + \min_u \{(F(x, t, u, v^1), \ S_x^1)\} = 0, S^1(x, \ T) = f(x), u \in P \quad (2.6)$$

As before, by assuming the existence and continuous differentiability of the function $S^1$, we find the first player's strategy $u^1(x, t)$ from the condition

$$\min_u \{(F(x, t, u, v^1), S_x^1)\} = H(x, t, u^1, v^1, S_x^1), \quad u \in P \qquad (2.7)$$

Strategy $u^1(x, t)$ is called the first player's approximate strategy in the first approximation. A new strategy $v^2(x, t)$ for the second player is found from the condition

$$\max_v \{(F(x, t, u^1, v), S_x^1)\} = H(x, t, u^1, v^2, S_x^1), \quad v \in Q_t \qquad (2.8)$$

Using relations (2.5) and (2.7) and arguing as in [5] (Theorem 1), we can show that the strategies $u^1(x, t)$ and $v^1(x, t)$ thus found are admissible in the sense defined in Sect. 1. Using (2.7) we write the boundary-value problem (2.6) as

$$S_t^1 + H(x, t, u^1, v^1, S_x^1) = 0, \quad S^1(x, T) = f(x) \qquad (2.9)$$

Let us estimate the difference between the Bellman function $S$ and function $S^1$. For this we consider the function

$$\alpha_1^\varepsilon(x, t) = H(x, t, u^1, v^1, S_x^1) - H(x, t, u^1, v^2, S_x^1) \leqslant 0$$

The inequality

$$|\alpha_1^\varepsilon| = |((F(x, t, u^1, v^1) - F(x, t, u^1, v^2)), S_x^1)| \leqslant \|F(x, t, u^1, v^1) - F(x, t, u^1, v^2)\| \|S_x^1\| \leqslant C' \|v^1 - v^2\| \|S_x^1\| \leqslant C'\varepsilon(1 + \|x\|^2)^m$$

is valid assuming that

$$\|S_x^1\| \leqslant C^2(1 + \|x\|^2)^m, \quad c, c^1, c^2 = \text{const} \qquad (2.10)$$

Therefore

$$0 \geqslant \alpha_1^\varepsilon \geqslant -C\varepsilon(1 + \|x\|^2)^m$$

The following result is valid.

Theorem 2.1. Let continuously differentiable solutions of problems (2.1) and (2.9) exist and let estimate (2.10) be fulfilled. Then the inequality

$$0 \leqslant S(x, t) - S^1(x, t) \leqslant (\lambda m)^{-1}\varepsilon C[e^{\lambda m(T-t_0)} - 1](1 + \|x\|^2)^m, \quad \lambda = \text{const} \qquad (2.11)$$

is valid.

Proof. Taking notation (2.2) into account, we write boundary-value problem (2.1) as

$$S_t + H(x, t, u^*, v^*, S_x) = 0, \quad S(x, T) = f(x) \qquad (2.12)$$

From the Lemma's result it follows that the strategy pair $(u^*(x, t), v^*(x, t))$ provides the saddle point of the game being examined. Therefore, the inequality

$$H(x, t, u^*, v^1, S_x) \leqslant H(x, t, u^*, v^*, S_x) \leqslant H(x, t, u^1, v^*, S_x) \qquad (2.13)$$

is valid. On the other hand, from the definition of strategy $v^2(x, t)$ in (2.8), allowing for the notation introduced for $\alpha_1^\varepsilon$, we obtain

$$H(x, t, u^1, v^2, S_x^1) = H(x, t, u^1, v^1, S_x^1) - \alpha_1^\varepsilon(x, t) \geqslant H(x, t, u^1, v^*, S_x^1) \qquad (2.14)$$

Further, the inequality

$$H(x, t, u^1, v^1, S_x^1) \leqslant H(x, t, u^*, v^1, S_x^1) \qquad (2.15)$$

follows from (2.7). Using the second of inequalities (2.13) and Eq. (2.12), we obtain

$$S_t + H(x, t, u^1, v^*, S_x) \geqslant 0, \quad S(x, T) = f(x) \tag{2.16}$$

The fulfillment of inequality

$$S_t^1 + H(x, t, u^1, v^*, S_x^1) \leqslant -\alpha_1^\varepsilon(x, t), \quad S^1(x, T) = f(x) \tag{2.17}$$

follows from inequality (2.14) and Eq. (2.9). Subtracting inequality (2.17) from (2.16), we obtain an inequality and a boundary condition for the function $Z = S - S^1$

$$Z_t + H(x, t, u^1, v^*, Z_x) \geqslant \alpha_1^\varepsilon(x, t), \quad Z(x, T) = 0 \tag{2.18}$$

This means (see [4], for exapmle) that for almost all $t \in [t_0, T]$ the inequality

$$dZ(x[t], t) / dt \geqslant \alpha_1^\varepsilon(x, t) \geqslant -C\varepsilon(1 + \|x[t]\|^2), \ Z(x, T) = 0 \tag{2.19}$$

is fulfilled for any motion $x[t] = x[t, x_0, t_0, u^1, v^*]$. Here the derivative is computed along the motion $x[t]$. The existence of this derivative for almost all $t \geqslant t_0$ follows from the continuous differentiability of functions $S$ and $S^1$ and from the absolute continuity of the motions [4].

Using inequality (1.2) and the original equation (1.1), we obtain the valid estimate

$$(1 + \|x[t]\|^2)^m \leqslant (1 + \|x_0\|^2)^m e^{m\lambda(t-t_0)}), \quad t \in [t_0, T] \tag{2.20}$$

Here $\lambda$ is the constant appearing in estimate (1.2). Integrating inequality (2.19) from $t_0$ to $T$ and allowing for (2.20) and the boundary condition $Z(x, T) = 0$, we obtain the inequality

$$S(x_0, t_0) - S^1(x_0, t_0) \leqslant (\lambda m)^{-1}\varepsilon C\,[e^{\lambda m(T-t_0)} - 1]\,(1 + \|x_0\|^2)^m \tag{2.21}$$

Using the first of inequalities (2.13) and Eq. (2.12), and next the inequality (2.15) and Eq. (2.9), we obtain

$$S_t + H(x, t, u^*, v^1, S_x) \leqslant 0, \quad S(x, T) = f(x)$$

$$S_t^1 + H(x, t, u^*, v^1, S_x^1) \geqslant 0, \quad S^1(x, T) = f(x)$$

Subtracting inequality (2.19) from (2.18), we obtain an inequality and a boundary condition for the function $Z^1 = S^1 - S$

$$dZ^1(x[t], t) / dt \geqslant 0, \quad Z^1(x, T) = 0 \tag{2.22}$$

for any motion $x[t] = x[t, x_0, u^*, v^1]$. Analogously to (2.21) the inequality $S^1 - S \leqslant 0$ follows from (2.22). We obtain inequality (2.11) by taking (2.21) into account.

Corollary. If $\alpha_1^\varepsilon(x, t) \equiv 0$, the strategy pair $(u^1(x, t), v^1(x, t))$ provides the saddle point of the game being examined.

Proof. In this case it follows from inequality (2.11) that $S = S^1$. Therefore, the equalities

$$S_t + H(x, t, u^1, v^1, S_x) = 0, \quad S(x, T) = f(x)$$

are valid. By virtue of the Lemma's result the latter means that the strategy pair $(u^1(x, t), v^1(x, t))$ provides the saddle point of the game being examined.

Note 1. Inequality (2.11) shows to what extent the value of the Bellman function $S$ at point $(x, t)$ differs from the minimal value of the pay-off functional, which can be achieved by the first player under the initial conditions $x = x_0$, $t = t_0$ in problem (1.1) when the second player applies the strategy $v^1(x, t)$ defined in (2.5).

Note 2. We can consider another iteration process. For this we select some strategy $u^\circ(x, t)$. (As $u^\circ(x, t)$ we can take, for example, the strategy obtained at the first step of the iteration process considered earlier).

Consider the boundary-value problem

$$W_t^\circ + \max_v \{(F(x, t, u^\circ, v), W_x^\circ)\} = 0, \quad W^\circ(x, T) = f(x)$$
$$v \in Q_\varepsilon$$

We find a strategy $v^\circ(x, t)$ from the condition

$$\max_v \{(F(x, t, u^\circ, v), W_x^\circ)\} = H(x, t, u^\circ, v^\circ, W_x^\circ), \quad v \in Q_\varepsilon$$

We find a new strategy $u^1(x, t)$ of the first player from the condition

$$\min_u \{(F(x, t, u, v^\circ), W_x^\circ)\} = H(x, t, u^1, v^\circ, W_x^\circ), \quad u \in P$$

We find the function $W^1$ and the strategies $v^1$ and $u^2$ analogously. Consider the function

$$\beta_1^\varepsilon(x, t) = H(x, t, u^1, v^1, W_x^1) - H(x, t, u^2, v^1, W_x^1)$$

It is clear that $\beta_1^\varepsilon(x, t) \geqslant 0$. Assume the validity of the estimate

$$\beta_1^\varepsilon \leqslant K(\varepsilon)(1 + \|x\|^2)^m, \quad K(\varepsilon) \geqslant 0$$

From the construction of the iteration process it is clear that $K(\varepsilon) = 0$ when $\varepsilon = 0$. However, in this case it is impossible to guarantee that $K(\varepsilon) \to 0$ as $\varepsilon \to 0$. Using the same arguments as in the proof of Theorem 2.1 we can prove the validity of the inequality

$$-(\lambda m)^{-1} K(\varepsilon)\, [e^{\lambda m(T-t_0)} - 1]\, (1 + \|x\|^2)^m \leqslant S - W^1 \leqslant 0$$

This estimate shows to what extent the value of the Bellman function $S$ at point $(x, t)$ differs from the maximal value of the pay-off functional, which can be achieved by the second player under the initial conditions $x = x_0$, $t = t_0$ in problem (1.1) when the first player applies the strategy $u^1(x, t)$ obtained at the first step of the iteration process described. As we noted earlier, the effectiveness of this estimate is small because $K(\varepsilon)$ may not tend to zero as $\varepsilon \to 0$.

Note 3. The conditions of continuous differentiability of functions $S$ and $W$ can be weakened if we take advantage of the result of Theorem 2.1 of [6].

**3. Small-parameter method.** We assume that the set $Q_\varepsilon$ considered in Sect. 1 is a sphere of radius $\varepsilon$ in an $r$-dimensional space. Here $\varepsilon$ is a sufficiently small number. Problem (1.1) is then reduced to the form

$$dx/dt = F(x, t, u, \varepsilon v), \quad u \in P, \quad v \in Q_1$$

Here $Q_1$ is the unit sphere in the $r$-dimensional space. For solving this problem we apply the method proposed in [6–9].

We consider the minimax problem assuming the existence of a twice continuously differentiable solution of the boundary-value problem

$$S_t + H(x, t, S_x; \varepsilon) = 0, \qquad S(x, T) = f(x) \tag{3.1}$$

$$H(x, t, S_x; \varepsilon) = \min_u \max_v \{(F(x, t, u, \varepsilon, v), S_x)\}, \qquad u \in P \tag{3.2}$$
$$v \in Q_1$$

According to the result of Theorem 1 from [4], the set of vectors $u^*(x, t)$ providing the minimum in (3.2) determines the first player's minimax strategy.

Let the condition be fulfilled:

1) the function $H(x, t, S_x; \varepsilon)$ is continuous together with its derivatives with respect to $S_x$ and $\varepsilon$ up to second order, inclusive. We seek the solution of problem (3.1) as an expansion in powers of parameter $\varepsilon$

$$S(x, t) = S^\circ(x, t) + \varepsilon S^1(x, t) + \ldots \tag{3.3}$$

We represent the function $H$ in the form

$$\begin{gathered} H(x, t, S_x; \varepsilon) = H(x, t, S_x^\circ; 0) + \varepsilon H_\varepsilon(x, t, S_x^\circ; 0) + \\ \varepsilon(\nabla H(x, t, S_x^\circ; 0), S_x^1) + \ldots \\ H_\varepsilon = \frac{\partial}{\partial\varepsilon} H(x, t, S_x^\circ; \varepsilon) \end{gathered} \tag{3.4}$$

Here $\nabla H$ is the vector of partial derivatives with respect to the components of vector $S_x$. Substituting expansion (3.3) into (3.1) and limiting ourselves only to terms of first order in $\varepsilon$, we find that function $S^\circ$ is the solution of the boundary-value problem

$$S_t^\circ + H(x, t, S_x^\circ; 0) = 0, \quad S^\circ(x, T) = f(x) \tag{3.5}$$

Here we assume that

2) the solution of boundary-value problem (3.5) exists and has modulus-bounded continuous derivatives in $x$ up to second order, inclusive.

The function $S^1$ from expansion (3.3) is the solution of the problem

$$S_t^1 + (\nabla H(x, t, S_x^\circ; 0), S_x^1) + H_\varepsilon(x, t, S_x^\circ; 0) = 0, S^1(x, T) = 0 \tag{3.6}$$

The first-approximation equation (3.5) is the Bellman equation for the original problem which for $\varepsilon = 0$ can be treated as an optimal control problem. We shall assume that this problem is solved, i.e. we shall find the synthesis of the optimal control $u^\circ(x, t)$, the function $S^\circ(x, t)$ , and the correponding field of optimal trajectories

$$x = \psi(t, y) \tag{3.7}$$

Here $\psi(t, y)$ is a vector-valued function, $y$ is an $n$-dimensional vector of arbitrary constants. In addition to the assumptions already made we assume the fulfillment of the following condition:

3) equality (3.7) can be solved relative to $y$, i.e. we can obtain the relation

$$y = \varphi(x, t) \tag{3.8}$$

In order to solve the boundary-value problem (3.6) defining the second approximation we write out the system of equations determining the characteristics of Eq. (3.6)

$$dx/dt = -\nabla H(x, t, S_x^\circ; 0), \quad dS^1/dt = -H_\varepsilon(x, t, S_x^\circ; 0) \tag{3.9}$$

Taking into account the notation (3.2) introduced and the known equality $S_x^\circ = -p$ ($p$ is the vector of adjoint variables for the original system when $\varepsilon = 0$), we note that the first equation in (3.9) defines, according to the maximum principle, the optimal trajectories of the original system with $\varepsilon = 0$. The solution of the first equation from (3.9) is given by equality (3.7), while the system of first integrals is given by equality (3.8). As follows from (3.7)—(3.9), the general solution of Cauchy problem (3.6) is determined by the expression

$$S^1(x, t) = -\int_{t_0}^{t} H_\varepsilon(\xi, \tau, S_\xi^\circ(\xi, \tau); 0)\, d\tau \tag{3.10}$$

$$\xi = \psi(t, \varphi(x, t)) \tag{3.11}$$

The first and second players' strategies in the first approximation are found from the condition

$$\min_u \max_v \{(F(x, t, u, \varepsilon v), \quad S_x^\circ + \varepsilon S^1)\} = (F(x, t, u^1, \varepsilon v^1), \tag{3.12}$$

$$S_x^\circ + \varepsilon S_x^1) = H(x, t, S_x^\circ + \varepsilon S_x^1; \varepsilon), \quad u \in P, \ v \in Q_1$$

Equalities (3.3) and (3.7)—(3.12) determine explicitly the approximate solution of the Bellman Eq. (3.1) and certain players' strategies if the solution of the corresponding optimal control problem is known. In the next theorem we have indicated the error bounds for the functions $S^\circ$ and $S^\circ + \varepsilon S^1$.

Theorem 3.1. Let conditions (1)—(3) be fulfilled. Then the bounds

$$| S - S^\circ | \leqslant C_1 \varepsilon, \quad | S - S^\circ - \varepsilon S^1 | \leqslant C_2 \varepsilon^2 \tag{3.13}$$

with certain constants $C_k$ $(k = 1, 2)$ are valid for $t \in [t_0, T]$ and $x \in R^n$.

Proof. From condition (1) and equality (3.3) follows the validity of the expansions

$$H(x, t, S_x; \varepsilon) = H(x, t, S_x^\circ; 0) + \varepsilon H_\varepsilon(x, t, S_x^\circ; \varepsilon_1) + \tag{3.14}$$
$$(\nabla H(x, t, M_1; \varepsilon), S_x - S_x^\circ)$$

$$H(x, t, S_x; \varepsilon) = H(x, t, S_x^\circ; 0) + (\nabla H(x, t, S_x^\circ; 0),$$

$$S_x - S_x^\circ) + \varepsilon H_\varepsilon(x, t, S_x^\circ; 0) + \varepsilon(\nabla H_\varepsilon(x, t, S_x^\circ; \varepsilon_2),$$

$$S_x - S_x^\circ) + (N(x, t, M_2; \varepsilon)(S_x - S_x^\circ), S_x - S_x^\circ) +$$
$$\varepsilon^2 H_{\varepsilon\varepsilon}(x, t, S_x^\circ; \varepsilon_3)$$

Here

$$M_i = S_x^\circ + \Theta_i (S_x - S_x^\circ), \quad 0 < \Theta_i < 1, \quad i = 1, 2,$$

$$0 < \varepsilon_k < \varepsilon, \quad k = 1, 2, 3$$

Using expansions (3.14), from (3.1) and (3.5) we find that the function $Z = S - S^\circ$ satisfies the boundary-value problem

$$Z_t + (\nabla H(x, t, M_1; \varepsilon), Z_x) = -\varepsilon H_\varepsilon(x, t, S_x^\circ; \varepsilon_1) \tag{3.15}$$

$$Z(x, T) = 0$$

Hence, allowing for the boundedness of the function $H_\varepsilon$, we obtain the first of inequalities (3.13).

Let us now show that the bound

$$| S_x - S_x^\circ | \leqslant C' \varepsilon, \quad C' = \text{const} \tag{3.16}$$

is valid. Using conditions (1) and (2) we differentiate Eq. (3.15) with respect to the variable $x_i$ and we denote $w^i = Z_{x_i}$. We obtain the boundary-value problem for the system of equations

$$w_t^i + (\nabla H_{x_i}(x, t, M_1; \varepsilon), w) + (\nabla H(x, t, M_1; \varepsilon), w_x^i) = \tag{3.17}$$

$$\varepsilon \frac{\partial}{\partial x_i} H_\varepsilon (x, t, S_x^\circ; \varepsilon_1)$$

$$w^i (x, T) = 0, \quad i = 1, \ldots, n; \quad w = (w^1, \ldots, w^n)$$

Equation system (3. 17) is linear; therefore, because the functions $\partial H_\varepsilon / \partial x_i$ are bounded, we obtain the estimates

$$| w^i | \leqslant C^i \varepsilon, \quad i = 1, \ldots, n$$

with certain constants $C^i$. Hence follows inequality (3. 16). Using expansions (3. 14) and Eqs. (3. 5), (3. 6) and (3. 1) we obtain the boundary-value problem for the function $Z^1 = S - S^\circ - \varepsilon S^1$

$$Z_t^1 + (\nabla H (x, t, S_x^\circ; 0), Z_x^1) = - \varepsilon^2 H_{\varepsilon\varepsilon} (x, t, S_x^\circ; \varepsilon_3) - \varepsilon (\nabla H (x, t, S_x^\circ; \varepsilon_2), S_x - S_x^\circ) + (N (x, t; M_2; \varepsilon) \times (S_x - S_x^\circ), S_x - S_x^\circ), \; Z^1 (x, T) = 0$$

Taking into account the proved inequality (3. 16), we find that the right-hand side of the last equation is a quantity of order $O(\varepsilon^2)$; consequently, the second bound in (3. 13) is valid with some constant $C_2$.

By $W (x, t)$ we denote the solution of the boundary-value problem

$$W_t + (F (x, t, u^1, \varepsilon v^1), W_x) = 0, \quad W (x, T) = f (x) \tag{3.18}$$

Here $u^1 (x, t)$ and $v^1 (x, t)$ are the first and second players' strategies in the first approximation, found from conditions (3. 12).

Theorem 3. 2. Let conditions (1) –(3) be fulfilled. Then the estimate

$$| S - W | \leqslant C\varepsilon^2 \tag{3.19}$$

is valid for function $W$

Proof. We denote $E = W - S^\circ - \varepsilon S^1$. From (3. 18), (3. 5) and (3. 6), allowing for notation (3. 12), we find that the right-hand side of the last equation is a quantity of the order of $\varepsilon^2$. Therefore, the inequality

$$| W - S^\circ - \varepsilon S^1 | \leqslant C_3 \varepsilon^2, \quad C_3 = \text{const} \tag{3.20}$$

is fulfilled with some constant. Using inequality (3. 20), and the inequality (3. 13) proved in Theorem 3. 1, we obtain the validity of (3. 19).

Let us consider examples illustrating the constructions made.

1°. Let the motion of a conflict-controlled object be described by the equation

$$d^2y/dt^2 = u (1 + v), \quad y (0) = y_0, \quad y^\cdot (0) = y_0^\cdot, \; t \in [0, T]$$

Here $y$ is a scalar and the controls take the values $| u | \leqslant 1$ and $| v | \leqslant \varepsilon < 1$. As the pay-off we examine the quantity $[y (T)]^2$. We introduce a new variable $x = y^\cdot (T - t) + y$. The original equation takes the form

$$dx/dt = (T - t) u (1 + v), \quad x (0) = x_0$$

Under such a change of variable the pay-off functional preserves its form, namely, $[x (T)]^2$. Let us construct the first and second players' approximate strategies, using the results of Sect. 2. We set $v^\circ \equiv 0$. The function $S^\circ$ and the strategy $u^\circ$ are determined from the solution of the boundary-value problem

$$S_\tau^\circ = \tau \min_{|u| \leqslant 1} \{u S_x^\circ\} = - \tau | S_x^\circ |, \quad S^\circ (x, 0) = x^2$$

Here $T - t = \tau$ is reverse time, $u^\circ = -\text{sign}\, S_x^\circ$. The solution of this problem is

$$S^\circ = \begin{cases} (|x| - \tau^2/2)^2, & |x| \geqslant \tau^2/2 \\ 0 & , \ |x| < \tau^2/2 \end{cases}$$

Strategy $u^\circ$ is uniquely defined in the region $|x| \geqslant \tau^2/2$, where $u^\circ = -\text{sign}\, x$, and nonuniquely in the remaining part of the region. Strategy $v^1$ is found from the condition

$$\max_{|v| \leqslant \varepsilon} \{u^\circ (1 + v) S_x^\circ\}$$

whence it follows that $v^1 = -\varepsilon$ in region $|x| \geqslant \tau^2/2$. Strategy $v^1$ is determined ambiguously in the region $|x| < \tau^2/2$. Accordong to Sect. 2 strategy $v^1$ is the second player's approximate optimal strategy in the first approximation. We complete the definition of strategy $v^1$ in the region $|x| < \tau^2/2$ by setting $v^1 = -\varepsilon$ and we find the function $S^1$ and the strategy $u^1$. It can be verified that the function $\alpha_1^\varepsilon(x, t)$ is such that

$$\alpha_1^\varepsilon(x, t) = u^1(1 + v^1) S_x^1 - u^1(1 + v^2) S_x^1 \equiv 0$$

By virtue of inequality (2.11) $S^1 = S$ is the Bellman function of the problem being analyzed. According to the Corollary the strategy pair $(u^1(x, t),\ v^1(x, t))$ provides the saddle point.

2°. Let us consider a planar controlled motion with a gap

$$dx_1/dt = x_3, \quad dx_3/dt = -x_3 + u_1 \cos v - u_2 \sin v$$

$$dx_2/dt = x_4, \quad dx_4/dt = -x_4 + u_1 \sin v + u_2 \cos v$$

$$t \in [0, T], \quad x_i(0) = x_i^\circ, \quad i = 1, 2, 3, 4$$

The first player's aim is to choose a control $u$, subject to the constraint $u_1^2 + u_2^2 \leqslant 1$, so as to minimize the value of a functional of the terminal state

$$J = \sqrt{x_1^2(T) + x_2^2(T)}$$

The second player's aim is to minimize the value of functional $J$ by choosing $v$ (the gap) such that $|v| \leqslant \varepsilon < \pi/2$. We shall solve the minimax problem by applying the small-parameter method presented in Sect. 3, and assuming $\varepsilon$ to be a fairly small number. We seek the solution of the Bellman equation in the form $S = S^\circ + \varepsilon S^1 + \ldots$ The boundary-value problem for the function $S^\circ$, being the first approximation, is

$$S_\tau^\circ = x_3 S_{x_1}^\circ + x_4 S_{x_2}^\circ - x_3 S^\circ_{x_3} - x_4 S_{x_4}^\circ + \min_{u_1^2 + u_2^2 \leqslant 1} \{u_1 S_{x_3}^\circ + u_2 S_{x_4}^\circ\}$$

$$S^\circ(x, 0) = \sqrt{x_1^2 + x_2^2}$$

Here $T - t = \tau$ is reverse time. Computing the minimum, we obtain

$$\min_{u_1^2 + u_2^2 \leqslant 1} \{u_1 S_{x_3}^\circ + u_2 S_{x_4}^\circ\} = -\sqrt{[S_{x_3}^\circ]^2 + [S_{x_4}^\circ]^2} = R$$

achieved on a vector $u^\circ$ with components

$$u_1^\circ = S^\circ_{x_3} / R, \quad u_2^\circ = S^\circ_{x_4} / R$$

The boundary-value problem for the second approximation is

$$S_\tau^1 = x_3 S^1_{x_1} + x_4 S^1_{x_2} - x_3 S^1_{x_3} - x_4 S^1_{x_4} + u_1^\circ S^1_{x_3} + u_2^\circ S^1_{x_4}$$

$$S^1(x, 0) = 0$$

From the uniqueness of the solution of the Cauchy problem for first-order hyperbolic partial differential equation it follows that $S^1(x, t) \equiv 0$. By direct verification it is easy to see that the function $S^\circ$, satisfying the boundary-value problem for the first approximation, is given by the expression

$$S^\circ(x, t) = \{[x_1 + (1 - e^{-\tau})x_3]^2 + [x_2 + (1 - e^{-\tau})x_4]^2\}^{1/2} + (e^{-\tau} + \tau) - 1$$

Hence it follows that

$$u_1^\circ = -\frac{x_1 - (1 - e^{-\tau})x_3}{W}, \quad u_2^\circ = -\frac{x_2 + (1 - e^{-\tau})x_4}{W}$$

$$W = \{[x_1 + (1 - e^{-\tau})x_3]^2 + [x_2 + (1 - e)^{-\tau}x_4]^2\}^{1/2}$$

We find the approximate minimax strategy and the second player's strategy from the condition (3.11)

$$\min_u \max_v \{\cos v\,(u_1 S_{x_3}^\circ + u_2 S_{x_4}^\circ) + \sin v\,(u_1 S_{x_4}^\circ - u_2 S_{x_3}^\circ)\} =$$
$$\min_u \max_v \{\sqrt{[S_{x_3}^\circ]^2 + [S_{x_4}^\circ]^2 (u_1^2 + u_2^2)}\, \sin(v + \alpha)\}$$

Here $\alpha$ is an angle such that

$$\alpha = \operatorname{arctg}\left[\frac{u_1 S_{x_3}^\circ - u_2 S_{x_4}^\circ}{u_1 S_{x_4}^\circ - u_2 S_{x_3}^\circ}\right]$$

If $-\pi/2 \leqslant \alpha \leqslant \pi/2 - \varepsilon$, then the minimax in the expression written above is achieved for $v^1 = \varepsilon$ and $u^1 = u^\circ$. However, if $\pi/2 - \varepsilon < \alpha \leqslant \pi/2$, then $v^1 = \pi/2 - \alpha$ and $u^1 = 0$. From the physical sense of the problem it follows that the second case is realized when the original system arrives at the point $(x_1 = 0, x_2 = 0)$ with $u = 0$. The set of such points is given by the equations

$$x_1 = (e^{-\tau} - 1)x_3, \quad x_2 = (e^{-\tau} - 1)x_4$$

Taking this circumstance into account, we get that the approximate minimax strategy is $u^1 = u^\circ$. From Theorem 3.2 follows the assertion that for each fixed $x$ the solution $S^\circ$ differs from the true solution of the Bellman equation by a quantity of order $O(\varepsilon^2)$.

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# THE NONLINEAR PROBLEM OF UNSTEADY FILTRATION OF HEAVY FLUID WITH A FREE SURFACE

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A solution is derived for the problem of unsteady motion of heavy fluid with a free surface in the vertical plane in a porous medium. Such problems are encountered in irrigation and land improvement schemes in connection with the filtration of ground waters. To use numerical and approximate methods for obtaining a solution of this fairly difficult problem one must be sure of its existence. The case when the heavy fluid occupies, at the initial instant of time, a finite region, is considered. An earlier investigation of this problem by the author [1] was based on some other assumptions with the heavy fluid occupying a semi-infinite region.

Let region $L$ occupied by a heavy fluid be mapped onto a unit circle in plane $\zeta$ by means of function $z(\zeta, t)$, where the time $t$ is a parameter. At the initial instant of time

$$z(\zeta, 0) = z_0(\zeta) \tag{1}$$

In this representation the coordinate origin in the $\zeta$-plane corresponds to a drain in the $L$ region. Function $z(\zeta, t)$ which depends on the complex variable $\zeta$ and on time $t$ must satisfy some boundary condition at subsequent instants.

The velocity potential of the motion of a heavy fluid is

$$\varphi = -k\left(\frac{p}{\rho g} + y\right) \tag{2}$$

where $p$ is the pressure, $k$ is the filtration coefficient, $\rho$ is the density, and $g$ is the acceleration of gravity. The velocity components are

$$v_x = \frac{\partial\varphi}{\partial x}, \quad v_y = \frac{\partial\varphi}{\partial y}$$